NONCOMMUTATIVE POSITIVSTELLENSÄTZE FOR PAIRS REPRESENTATION-VECTOR

JAKA CIMPRIČ

Abstract. We study non-commutative real algebraic geometry for a unital associative ∗-algebra \(A\) viewing the points as pairs \((\pi, v)\) where \(\pi\) is an unbounded ∗-representation of \(A\) on an inner product space which contains the vector \(v\). We first consider the ∗-algebras of matrices of usual and free real multivariate polynomials with their natural subsets of points. If all points are allowed then we can obtain results for general \(A\). Finally, we compare our results with their analogues in the usual (i.e. Schmüdgen’s) non-commutative real algebraic geometry where the points are unbounded ∗-representation of \(A\).

1. Introduction

Classical real algebraic geometry is interested in positivity sets of real multivariate polynomials, i.e. sets of the form \(K_S = \{a \in \mathbb{R}^d \mid p(a) \geq 0 \text{ for every } p \in S\}\) where \(S\) is a finite subset of \(\mathcal{P}_d := \mathbb{R}[X_1, \ldots, X_d]\). The main question is to compute the set \(\text{Sat}_{\geq}(S) = \{q \in \mathcal{P}_d \mid q(a) \geq 0 \text{ for all } a \in K_S\}\) of all nonnegative polynomials on \(K_S\) and the set \(\text{Sat}_{>}(S) = \{q \in \mathcal{P}_d \mid q(a) > 0 \text{ for all } a \in K_S\}\) of all positive polynomials on \(K_S\). The answer is given by Stengle’s Positivstellensatz, see [15, Prop. 2.2.1]. Better answers are known under various compactness assumptions, see e.g. Jacobi’s representation theorem [15, Theorem 5.4.4]. We are interested in noncommutative generalizations of this theory.

Our first result, Theorem 2.1, extends the classical theory to matrix polynomials. Fix \(n \in \mathbb{N}\) and write \(M_n(\mathcal{P}_d)\) (resp. \(S_n(\mathcal{P}_d)\)) for the set

Date: July 20th 2010, revised October 5th 2010.

2010 Mathematics Subject Classification. 14A22, 14P10, 16W10, 16W80, 13J30.

Key words and phrases. positive polynomials, real algebraic geometry, matrix polynomials, free algebras, algebras with involution.
of all (resp. all symmetric) $n \times n$ matrices with entries from the set \( \mathcal{P}_d \). Write \( \Sigma_n \) for the set of all $n \times n$ real positive semidefinite matrices.

Let us define the positivity set of a finite subset \( S \subseteq S_n(\mathcal{P}_d) \) by
\[
K'_S = \{(a, B) \mid a \in \mathbb{R}^d, B \in \Sigma_n \setminus \{0\} \text{ and } \text{Tr}(p(a)B) \geq 0 \text{ for all } p \in S\}.
\]

Theorem 2.1 computes the set \( \text{Sat}'_{\geq}(S) = \{q \in S_n(\mathcal{P}_d) \mid \text{Tr}(q(a)B) > 0 \text{ for all } (a, B) \in K'_S\} \)
under the additional assumption that \( S \) contains an element of the form \( (K_2 - \sum_{i=1}^d X_i^2)I_n \) where \( K \) is a nonzero real number and \( I_n \) is the identity matrix of size \( n \).

Note that this result is a variant of a theorem of Scherer and Hol (see [14, Theorem 13] which extends \([19, \text{Theorem 2}]\); a more general result follows from \([1, \text{Theorem 3 and Lemma 5}]\)), where the positivity set of \( S \) is defined by
\[
K'_{hs} = \{a \in \mathbb{R}^d \mid p(a) \text{ is positive semidefinite for all } p \in S\}
\]
and the question is to compute the set \( \text{Sat}_{\geq}^{hs}(S) = \{q \in S_n(\mathcal{P}_d) \mid q(a) \text{ is positive definite for all } a \in K'_{hs}\} \).

For \( n = 1 \) both results reduce to Jacobi’s representation theorem, see \([15, \text{Theorem 5.4.4}]\).

Our second result, Theorem 3.1, extends the classical theory to free \(*\)-polynomials, i.e. elements of the free algebra
\[
F_d = \mathbb{R}\langle X_1, \ldots, X_d, Y_1, \ldots, Y_d \rangle
\]
with the involution defined by
\[
Y_1^* = X_1, \ldots, Y_d^* = X_d.
\]
It also covers matrix versions of such polynomials, i.e. elements of the algebra \( M_n(F_d) \) with involution \([f_{ij}]^* = [f_{ji}^*]\). Let \( S \) be a finite subset of \( S_n(F_d) = \{f \in M_n(F_d) \mid f^* = f\} \). Its positivity set is
\[
K''_S = \{(A_1, \ldots, A_d, B) \mid \exists m \in \mathbb{N}: A_1, \ldots, A_d \in M_m, B \in \Sigma_{mn} \setminus \{0\} \text{ and } \text{Tr}(f(A_1, \ldots, A_d, A_1^T, \ldots, A_d^T)B) \geq 0 \text{ for all } f \in S\}
\]
where \( M_m \) is the set of all real \( m \times m \) matrices and the evaluations are performed entrywise. Theorem 3.1 computes the set
\[
\text{Sat}''_{\geq}(S) = \{f \in S_n(F_d) \mid \text{Tr}(f(A_1, \ldots, A_d, A_1^T, \ldots, A_d^T)B) \geq 0 \text{ for all } (A_1, \ldots, A_d, B) \in K''_S\}.
\]
For \( S = \emptyset \), it extends the main theorem of \([16]\) (which extends \([9]\)).
The closest result in literature is probably a theorem of Helton and McCullough, see [10, Theorem 1.2]. Here, the positivity set of $S$ is

$$K^\text{hm}_S = \{ (A_1, \ldots, A_d) \in B(H)^d \mid f(A_1, \ldots, A_d, A_1^*, \ldots, A_d^*) \text{ is positive definite for all } f \in S \},$$

where $B(H)$ is the set of all bounded operators on a real separable Hilbert space $H$ and evaluations are performed entrywise. The set

$$\text{Sat}_{\text{hm}}^>(S) = \{ f \in S_n(F_d) \mid f(A_1, \ldots, A_d, A_1^*, \ldots, A_d^*) \text{ is positive definite for all } (A_1, \ldots, A_d) \in K^\text{hm}_S \}$$

is computed under the additional assumption that $(K^2 - \sum_{i=1}^d X_i^* X_i)I_n$ belongs to $S$ for some nonzero real number $K$.

While the results of Scherer & Hol and Helton & McCullough fit into Schmüdgen’s approach to noncommutative real algebraic geometry [21] our results do not. In the last section we will formulate an alternative approach, motivated by a paper of Helton, McCullough and Putinar [12], and explain its relation to our results. The difference is in the definition of a noncommutative real point.

2. Matrix Polynomials

Let $n$ and $d$ be fixed natural numbers. Write $M_n(\mathcal{P}_d)$ for the algebra of all $n \times n$ matrices with entries from $\mathcal{P}_d = \mathbb{R}[X_1, \ldots, X_d]$, $S_n(\mathcal{P}_d)$ for the set of symmetric matrices from $M_n(\mathcal{P}_d)$ and $\Sigma_n(\mathcal{P}_d)$ for the set of all finite sums of elements of the form $P(X)^T P(X)$ where $P(X) \in M_n(\mathcal{P}_d)$. Write also $M_n$ (resp. $S_n$, $\Sigma_n$) for the set of all (resp. all symmetric, all positive semidefinite) $n \times n$ matrices with real entries. Let $I_n$ be the identity $n \times n$ matrix.

The aim of this section is to prove the following theorem:

**Theorem 2.1.** Pick

$$P_0(X), P_1(X), \ldots, P_k(X), Q(X) \in S_n(\mathcal{P}_d)$$

where $P_0(X) = (K^2 - \sum_{i=1}^d X_i^2)I_n$ for some nonzero real $K$. Suppose that for every $a \in \mathbb{R}^d$ and for every nonzero $B \in \Sigma_n$ such that

$$\text{Tr}(P_0(a)B) \geq 0, \text{Tr}(P_1(a)B) \geq 0, \ldots, \text{Tr}(P_k(a)B) \geq 0$$

we have that $\text{Tr}(Q(a)B) > 0$. Then, writing $T$ for all sums of squares of elements from $\mathcal{P}_d$, there exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$Q(X) - \varepsilon I_n \in \Sigma_n(\mathcal{P}_d) + T \cdot P_0(X) + T \cdot P_1(X) + \ldots + T \cdot P_k(X).$$

For $n = 1$ we get exactly Jacobi’s Representation Theorem [13, Theorem 5.4.4].
**Proof.** We say that a subset \( N \) of \( S_n(\mathcal{P}_d) \) is a weakly quadratic module (abbr. wqm) if

\[
N + N \subseteq N, \quad T \cdot N \subseteq N \quad \text{and} \quad \Sigma_n(\mathcal{P}_d) \subseteq N.
\]

The set \( \Sigma_n(\mathcal{P}_d) \) is the smallest wqm. Clearly, the set

\[
N_S := \Sigma_n(\mathcal{P}_d) + T \cdot P_0(X) + T \cdot P_1(X) + \ldots + T \cdot P_k(X)
\]

is the smallest wqm containing the set \( S = \{P_0(X), P_1(X), \ldots, P_k(X)\} \).

A wqm \( N \) is said to be *archimedean* if for every \( A \in S_n(\mathcal{P}_d) \) there exists a real \( r > 0 \) such that \( rI_n \pm A \in N \). In the sequel, we will abbreviate \( I_n \) to \( I \).

**Step 1.** The wqm \( N_S \) is archimedean.

Write \( B(N_S) = \{A \in S_n(\mathcal{P}_d) \mid \exists r \in \mathbb{R}^+: rI \pm A \in N_S\} \). Note that the set \( N_S \cap \mathcal{P}_d I \) is a quadratic module on \( \mathcal{P}_d I = \{pI \mid p \in \mathcal{P}_d\} \) which contains \( (K^2 - \sum_{i=1}^d X_i^2) I \) for some nonzero real \( K \). By [15, Corollary 5.2.4], \( N_S \cap \mathcal{P}_d I \) is archimedean in \( \mathcal{P}_d I \). It follows that \( B(N_S) \) contains \( \mathcal{P}_d I \). On the other hand, it is clear that \( S_n \subseteq B(N_S) \). Suppose now that \( p \in \mathcal{P}_d \) and \( A \in S_n \). Pick \( k, l \in \mathbb{R}^+ \) such that \( (k - p)I \in N_S \) and \( U \pm A \in N_S \). It follows \( klI \pm pA = l(k - p)I + p(U \pm A) \in N_S \), so that \( pA \in B(N_S) \). Clearly, \( B(N_S) \) is closed for addition. It follows that \( S_n(\mathcal{P}_d) \subseteq B(N_S) \), by decomposing each element of \( S_n(\mathcal{P}_d) \) as a sum of products of elements from \( T \) and \( S_n \). This proves the claim.

**Step 2.** If \( M \subseteq S_n(\mathcal{P}_d) \) is a wqm maximal subject to \( -I \not\in M \) and if \( M \) is archimedean, then \( M \cup -M = S_n(\mathcal{P}_d) \).

This is almost the same as [15, Theorem 5.2.5, part (1)]). If there exists \( A \in S_n(\mathcal{P}_d) \) such that \( A \not\in M \cup -M \), then \( M + TA \) and \( M - TA \) are wqm which strictly contain \( M \). By the maximality of \( M \), \( -I \in M + TA \) and \( -I \in M - TA \), so \( -I = B_1 + At_1 \) and \( -I = B_2 - At_2 \) for some \( t_1, t_2 \in T \) and \( B_1, B_2 \in M \). Multiplying the first equality by \( t_2 \) and the second by \( t_1 \) and adding them, we get \( -(t_2 + t_1)I = B_1t_2 + B_2t_1 \in M \). It follows that \( -t_1I \in M \). Now pick \( l \in \mathbb{R}^+ \) such that \( U + A \in M \). It follows that \( -I = B_1 + t_1(U + A) + l(-t_1I) \in M + TM + TM \subseteq M \). This is a contradiction with \( -I \not\in M \).

For every wqm \( N \) write \( \mathcal{K}_N \) for the set of all mappings \( \alpha: S_n(\mathcal{P}_d) \to \mathbb{R} \) such that

1. \( \alpha(N) \geq 0 \),
2. \( \alpha(I) = 1 \),
3. \( \alpha(A + B) = \alpha(A) + \alpha(B) \) for every \( A, B \in S_n(\mathcal{P}_d) \) and
4. \( \alpha(tA) = \alpha(tI)\alpha(A) \) for every \( t \in \mathcal{P}_d, A \in S_n(\mathcal{P}_d) \).
For every $\alpha \in \mathcal{K}_N$, the set $\alpha^{-1}(\mathbb{R}^+)$ is clearly an archimedean wqm containing $N$.

**Step 3.** If $N \subseteq S_n(\mathcal{P}_d)$ is an archimedean wqm then $-I \not\in N$ iff $\mathcal{K}_N \neq \emptyset$.

If there is an $\alpha \in \mathcal{K}_N$, then $\alpha(N) \geq 0$ and $\alpha(-I) = -1$, hence $-I \not\in N$. Conversely, if $-I \not\in N$, then there exists by Zorn’s Lemma a wqm $M$ containing $N$ and maximal subject to $-I \not\in M$. Since $N$ is archimedean, $M$ is also archimedean. By Step 2, $M \cup -M = S_n(\mathcal{P}_d)$, so we can define a mapping $\alpha: S_n(\mathcal{P}_d) \to \mathbb{R}$ by

$$\alpha(A) := \sup\{r \in \mathbb{R} \mid A - rI \in M\} = \inf\{r \in \mathbb{R} \mid rI - A \in M\}.$$ 

Clearly, $\alpha(N) \geq 0$ and $\alpha(I) = 1$. The same argument as in [15 Theorem 5.2.5, part (2)] shows that (3) is true for every $A, B \in S_n(\mathcal{P}_d)$ and that (4) is true for every $t \in T, A \in M$. Since $\alpha(-A) = -\alpha(A)$ for every $A \in M$ (by (3)) and since $T - T = \mathcal{P}_d$ and $M - M = S_n(\mathcal{P}_d)$, it follows that (4) is true for every $t \in \mathcal{P}_d$ and $A \in S_n(\mathcal{P}_d)$. Thus $\alpha \in \mathcal{K}_M$.

**Step 4.** A mapping $\alpha: S_n(\mathcal{P}_d) \to \mathbb{R}$ belongs to $\mathcal{K}_{S_n(\mathcal{P}_d)}$ iff there exists a matrix $B \in \Sigma_n$ with $\text{Tr}(B) = 1$ and a point $a \in \mathbb{R}^n$ such that $\alpha(P(X)) = \text{Tr}(P(a)B)$ for every $P(X) \in S_n(\mathcal{P}_d)$.

Clearly, the homomorphism $\alpha|_{\mathcal{P}_d}$ is the evaluation at the point $a = (\alpha(X_1I), \ldots, \alpha(X_dI)) \in \mathbb{R}^n$ and $\alpha|_{S_n}$ is a positive functional, hence of the form $A \mapsto \text{Tr}(AB)$ for some $B \in \Sigma_n$. Since $\alpha(I) = 1$, we have that $\text{Tr}(B) = 1$. By additivity, $\alpha(P(X)) = \text{Tr}(P(a)B)$ for every $P(X) \in S_n(\mathcal{P}_d)$. The converse is clear.

**Step 5.** Suppose that $N$ is an archimedean wqm and $A = A(X) \in S_n(\mathcal{P}_d)$ is such that $\alpha(A) > 0$ for every $\alpha \in \mathcal{K}_N$. Then $A \in \varepsilon I + N$ for some $\varepsilon > 0$.

The proof is the same as in [15 Theorem 5.4.4]. Write $N_1 = N - TA$. Since $N \subseteq N_1$, $N_1$ is archimedean. The assumption $\alpha(A) > 0$ for every $\alpha \in \mathcal{K}_N$ implies that $\mathcal{K}_{N_1} = \emptyset$. By Step 3, $-I \not\in N_1$. Pick $S = S(X) \in N$ and $t \in T$ such that $-I = S - tA$, so $tA - I = S \in N$. Since $N$ is archimedean, there exists $k \in \mathbb{R}^+$ such that $(2k - 1)I - t^2A \in N$ and $(2k - t)I \in N$. Consider the identity: $k^2A + (k^2r - 1)I = (k-t)^2(A+rI)+2k(tA-I)+rt(2k-t)I+((2k-1)I-t^2A)$. This shows that $A+rI \in N$ implies that $A + (r - \frac{1}{k^2})I \in N$. Repeating this step several times, we eventually find an $\varepsilon > 0$ such that $A - \varepsilon I \in N$. 

We can extend Theorem 2.1 to the non-compact case.

**Theorem 2.2.** Suppose that $P_1(X), \ldots, P_k(X), Q(X) \in S_n(\mathcal{P}_d)$ are homogeneous polynomials of even degree such that for every $a \in \mathbb{R}^d \setminus \{0\}$
and for every $B \in \Sigma_n \setminus \{0\}$ which satisfy

$$
\text{Tr}(P_1(a)B) \geq 0, \ldots, \text{Tr}(P_k(a)B) \geq 0
$$

we have that $\text{Tr}(Q(a)B) > 0$. Then there exists $m \in \mathbb{N}$ such that

$$
\left(\sum_{i=1}^{d} X_i^2\right)^m Q(X) \in \Sigma_n(\mathcal{P}_d) + \sum_{j=1}^{k} TP_j(X).
$$

A variant of this result (corresponding to Schmüdgen’s approach) is proved in [1]. For $n = 1$ both results reduce to a theorem of Putinar and Vasilescu, see [18].

**Proof.** Write $P_0(X) = (1 - \sum_{i=1}^{d} X_i^2) \cdot 1$. From the assumption, it follows that $\text{Tr}(Q(a)B) > 0$ for every point $a \in \mathbb{R}^n$ and for every nonzero $B \in \Sigma_n$ such that

$$
\text{Tr}(P_0(a)B) = 0, \text{Tr}(P_1(a)B) \geq 0, \ldots, \text{Tr}(P_k(a)B) \geq 0.
$$

By Theorem [2.1] there exist $q_0(X) \in \mathcal{P}_d$, $q_1(X), \ldots, q_k(X) \in T$ and $S(X) \in \Sigma_n(\mathcal{P}_d)$ such that

$$
Q(X) = S(X) + q_0(X)P_0(X) + q_1(X)P_1(X) + \ldots + q_k(X)P_k(X).
$$

From now on we compute in the localization of $\mathcal{P}_d$ by $\|X\| = \sqrt{\sum X_i^2}$. Every elements of this localization can be written uniquely as

$$
g(X) + h(X)\|X\|
$$

where $g(X), h(X) \in \mathcal{P}_d$ and $l \in \mathbb{N}$. Since $P_0(\frac{X}{\|X\|}) = 0$, we get

$$
Q(\frac{X}{\|X\|}) = S(\frac{X}{\|X\|}) + \sum_{j=1}^{k} q_j(\frac{X}{\|X\|})P_j(\frac{X}{\|X\|}).
$$

By clearing denominators and comparing components at $1$ and $\|X\|$, we get (because $Q(X)$ and $P_j(X)$ are homogeneous of even degree) that

$$
\|X\|^{2m} Q(X) \in \Sigma_n(\mathcal{P}_d) + \sum_{j=1}^{k} TP_j(X)
$$

for some $m \in \mathbb{N}$. \qed
3. Matrix free ∗-polynomials

Let \( n \) and \( d \) be fixed natural numbers. Write \( F_d \) for the free ∗-algebra in \( d \) variables as defined in the introduction and \( M_n(F_d) \) for the ∗-algebra of \( n \times n \) matrices with entries from \( F_d \) with involution defined by \([f_{ij}]^\ast = [f_{ji}]^\ast\). Write \( S_n(F_d) \) for the real vector space of all elements \( f \in M_n(F_d) \) such that \( f^\ast = f \) and \( \Sigma_n(F_d) \) for the set of all finite sums of elements of the form \( f^\ast f, \ f \in M_n(F_d) \). Let \( M_n, S_n \) and \( \Sigma_n \) be as in the previous section.

For every \( d \)-tuple \( C = (C_1, \ldots, C_d) \in (M_m)^d \), we define the mapping \( \text{ev}_C : F_d \to M_m \) by

\[
\text{ev}_C(f(X_1, \ldots, X_d, X_1^\ast, \ldots, X_d^\ast)) = f(C_1, \ldots, C_d, C_1^T, \ldots, C_d^T)
\]

and the mapping \( (\text{ev}_C)_n : M_n(F_d) \to M_{mn} \)

\[
(\text{ev}_C)_n([f_{ij}]) = [\text{ev}_C(f_{ij})].
\]

The aim of this section is to prove the following theorem.

**Theorem 3.1.** Pick \( n \) and \( d \). For every elements \( p_1, \ldots, p_k, q \in S_n(F_d) \), the following are equivalent:

1. \( \text{Tr}((\text{ev}_C)_n(q)B) \geq 0 \) for every \( m \in \mathbb{N}, \ C \in (M_m)^d \) and \( B \in \Sigma_{mn} \) such that \( \text{Tr}((\text{ev}_C)_n(p_i)B) \geq 0 \) for every \( i = 1, \ldots, k \).
2. \( q \in \Sigma_n(F_d) + \mathbb{R}^+ p_1 + \ldots + \mathbb{R}^+ p_k \), where the closure refers to the finest locally convex vector space topology of \( S_n(F_d) \).

The proof will depend on some rather elementary observations about ∗-representations that we are going to recall now. Recall that a ∗-representation of \( \mathcal{A} \) is an ordered pair \((\pi, D_\pi)\) where \( D_\pi \) is a real inner product space and \( \pi \) is a unital real algebra homomorphism from \( \mathcal{A} \) into the algebra of all linear operators on \( D_\pi \) such that \( \langle \pi(a)v_1, v_2 \rangle = \langle v_1, \pi(a^\ast)v_2 \rangle \) for every \( a \in \mathcal{A} \).

**Example 3.2.** For every \( m \in \mathbb{N} \) and \( C \in (M_m)^d \), the mapping \( \text{ev}_C : F_d \to M_m \) defines a ∗-representation \((\text{ev}_C, \mathbb{R}^m)\) of the ∗-algebra \( F_d \). Conversely, every ∗-representation \((\pi, D_\pi)\) of \( F_d \) for which \( m = \dim D_\pi < \infty \) comes from \( \text{ev}_C \) with \( C = (\pi(X_1), \ldots, \pi(X_d)) \in (M_m)^d \).

Let \( \mathcal{A} \) be a ∗-algebra and \((\pi, D_\pi)\) a ∗-representation of \( \mathcal{A} \). We can equip \( D_\pi \) with the structure of a left \( \mathcal{A} \)-module by setting \( av := \pi(a)v \) for every \( a \in \mathcal{A} \) and \( v \in D_\pi \). Equivalently, we can start with a left \( \mathcal{A} \)-module \( D \) equipped with a real valued inner product satisfying \( \langle av_1, v_2 \rangle = \langle v_1, a^\ast v_2 \rangle \) for every \( a \in \mathcal{A} \) and \( v_1, v_2 \in D \) and denote its action by \( \pi \). We say that the ∗-representations \((\pi, D_\pi)\) and \((\psi, D_\psi)\) are unitarily equivalent if there exist mutually inverse isometries \( S : D_\pi \to D_\psi \) and \( T : D_\psi \to D_\pi \) such that \( S\pi(a) = \psi(a)S \) and
$T\psi(a) = \pi(a)T$ for every $a \in A$. Equivalently, $S$ and $T$ are mutually inverse left $A$-module homomorphisms between $D_\pi$ and $D_\psi$ that preserve inner products.

**Lemma 3.3.** (GNS construction) Let $A$ be a $\ast$-algebra, Sym$(A) = \{a \in A \mid a = a^*\}$ and $\Sigma_A$ the set of all finite sums of elements $a^*a$, $a \in A$. Let $f$ be a real linear functional on Sym$(A)$ such that $f(\Sigma_A) \geq 0$. Then there exists a $\ast$-representation $(\pi_f, D_f)$ of $A$ and a vector $v_f \in D_f$ such that $f(a) = \langle \pi_f(a)v_f, v_f \rangle$ for every $a \in \text{Sym}(A)$.

**Proof.** The set $I_f = \{a \in A \mid f(a^*a) = 0\}$ is a left ideal in $A$ and $D_f := A/I_f$ is a left $A$-module with inner product $\langle x + I_f, y + I_f \rangle := \frac{1}{2}f(y^*x + x^*y)$. Let $\pi_f$ be the action of $D_f$ and $v_f := 1 + I_f \in D_f$. See Theorem 8.6.2 in [20] for details. \hfill \Box

**Lemma 3.4.** (Morita equivalence) Let $n$ be a natural number, $A$ a $\ast$-algebra and $M_n(A)$ the $\ast$-algebra of $n \times n$ matrices with entries in $A$ with involution $[a_{ij}]^* := [a_{ji}^*]$. Every $\ast$-representation $(\pi, D_\pi)$ of $A$ induces a $\ast$-representation $(\pi_n, D_{\pi_n})$ of $M_n(A)$ where $D_{\pi_n} := (D_{\pi})^n$ and $\pi_n([a_{ij}]) := [\pi(a_{ij})]$. Moreover, every $\ast$-representation of $M_n(A)$ is unitarily equivalent to a $\ast$-representation induced from $A$ as above.

**Proof.** The first part is clear. Write $E_{ij}$ for the element of $M_n(A)$ which has 1 at $(i, j)$-th place and zeros elsewhere. Pick a $\ast$-representation $(\psi, D_\psi)$ of $M_n(A)$ and note that $D := E_{11}D_\psi$ is a left $A$-submodule of $D_\psi$. We equip $D$ with the inner product inherited from $D_\psi$. Write $\pi$ be the action of $D$ and note that $D_{\pi_n} = D^n$. The mappings $v \mapsto (E_{11}v, \ldots, E_{1n}v)$ from $D_\psi$ to $D_{\pi_n}$ and $(v_1, \ldots, v_n) \mapsto E_{11}v_1 + \ldots + E_{nn}v_n$ from $D_{\pi_n}$ to $D_\psi$ are mutually inverse homomorphisms of left $M_n(A)$-modules which preserve inner products. \hfill \Box

The following is similar to Proposition 4 in [21]. For every element of $F_d$ we can define its degree as the total degree in $X_i$ and $X_j^\ast$. Write $F_{d,K}$ for the set of all elements of $F_d$ of degree $\leq K$.

**Lemma 3.5.** For every $\ast$-representation $(\pi, D_\pi)$ of $F_d$, every finite subset $\{v_1, \ldots, v_n\}$ of $D_\pi$ and every $K \in \mathbb{N}$ there exists a $\ast$-representation $(\rho, V)$ of $F_{d,K}$ where $V = \pi(F_{d,K})v_1 + \ldots + \pi(F_{d,K})v_n$ is a finite-dimensional subspace of $D_\pi$ containing $\{v_1, \ldots, v_n\}$ and $\rho(b)v_i = \pi(b)v_i$ for every $b \in F_{d,K}$ and $i = 1, \ldots, n$.

**Proof.** Let $H$ be the Hilbert space completion of $D_\pi$ and let $P$ be the orthogonal projection of $H$ to $V$. For every $x \in \{X_1, \ldots, X_d, X_1^\ast, \ldots, X_d^\ast\}$ write $\rho(x) = P\pi(x)|_V$. For every $u, u' \in V$, we have that $\langle \rho(x)u, u' \rangle_V = \langle \rho(x)u, u' \rangle_H = \langle P\pi(x)u, u' \rangle_H = \langle \pi(x)u, P^\ast u' \rangle_H = \langle \pi(x)u, u' \rangle_H$.
\( \langle u, \pi(x)^* u' \rangle_H = \langle u, \pi(x^*) u' \rangle_H = \langle P^* u, \pi(x)^* u' \rangle_H = \langle u, P \pi(x)^* u' \rangle_H = \langle u, \rho(x^*) u' \rangle_V \), hence \( \rho(x^*) = \rho(x)^* \). Since \( F_d \) is a free \*-*algebra, we can extend \( \rho \) to a \*-*representation of \( F_d \).

Fix \( i \) between 1 and \( n \). To prove that \( \rho(b) v_i = \pi(b) v_i \) for every \( b \in F_{d,K} \) we may assume that \( b \) is a monomial in \( X_i \) and \( X_j^* \) and proceed by induction on \( \deg b \). If \( b = 1 \), this is clear. Suppose that \( b = xc \) where \( b,c \in F_{d,K} \) and \( x \in \{ X_1, \ldots, X_d, X_1^*, \ldots, X_d^* \} \). By the inductive hypothesis, we have that \( \rho(c) v_i = \pi(c) v_i \). It follows that \( \rho(b) v_i = \rho(x) \rho(c) v_i = \rho(x) \pi(c) v_i \). Since \( \pi(c) v_i \in V \), we have that \( \rho(x) \pi(c) v_i = P \pi(x) (\pi(c) v_i) \) by the definition of \( \rho(x) \). Since \( b \in F_{d,K} \), we have that \( \pi(b) v_i \in V \), hence \( P \pi(x) (\pi(c) v_i) = P \pi(b) v_i = \pi(b) v_i \). Therefore, \( \rho(b) v_i = \pi(b) v_i \).

We are now able to give the proof of Theorem 3.1.

**Proof.** Recall that the finest locally convex topology of a real vector space is the topology whose fundamental system of neighbourhoods of zero consists of all convex absorbing sets. In particular, every real linear functional on \( S_n(F_d) \) is continuous in this topology. By the Separation theorem, an element \( q \) of \( S_n(F_d) \) belongs to the closure of a convex cone \( C \subseteq S_n(F_d) \) if and only if \( f(q) \geq 0 \) for every real linear functional \( f \) on \( S_n(F_d) \) such that \( f(C) \geq 0 \). For \( C = \Sigma_n(F_d) + \mathbb{R} p_1 + \ldots + \mathbb{R}^+ p_k \), we get that (2) is equivalent to

\[
(A) \quad f(q) \geq 0 \quad \text{for every real linear functional } f \text{ on } S_n(F_d) \quad \text{such that } \quad f(\Sigma_n(F_d)) \geq 0 \quad \text{and } \quad f(p_i) \geq 0 \quad \text{for every } i = 1, \ldots, k.
\]

For every \*-*representation \( \pi \) of \( A \) and for every \( v \in D_\pi \), the real linear functional \( f_{(\pi,v)}(a) := \langle \pi(a) v, v \rangle \), \( a \in S_n(F_d) \), satisfies \( f_{(\pi,v)}(\Sigma_n(F_d)) \geq 0 \). Conversely, for every real linear functional \( f \) on \( S_n(F_d) \) such that \( f(\Sigma_n(F_d)) \geq 0 \) there exist by Lemma 3.3 a \*-*representation \( (\pi_f, D_f) \) of \( M_n(F_d) \) and a vector \( v_f \in D_f \) such that \( f(a) = \langle \pi_f(a) v_f, v_f \rangle \) for every \( a \in S_n(F_d) \). It follows that (A) is equivalent to:

\( (B) \quad \langle \pi(q) v, v \rangle \geq 0 \quad \text{for every } \*-*\text{-representation } (\pi, D_\pi) \text{ of } M_n(F_d) \text{ and every } v \in D_\pi \text{ such that } \langle \pi(p_1) v, v \rangle \geq 0, \ldots, \langle \pi(p_k) v, v \rangle \geq 0. \)

By Lemma 3.4 every \*-*representation of \( M_n(F_d) \) is unitarily equivalent to a \*-*representation of the form \( \psi_n \). Therefore, (B) is equivalent to

\( (C) \quad \langle \psi_n(q) u, u \rangle \geq 0 \quad \text{for every } \*-*\text{-representation } (\psi, D_\psi) \text{ of } F_d \text{ and every } u \in (D_\psi)^n \text{ such that } \langle \psi_n(p_i) u, u \rangle \geq 0 \text{ for all } i = 1, \ldots, k. \)

For every \*-*representation of \( (\psi, D_\psi) \) of \( F_d \) and every \( u = (u_1, \ldots, u_n) \in (D_\psi)^n \), there exists by Lemma 3.5 a finite-dimensional \*-*representation \( (\rho, D_\rho) \) such that \( \phi(a) u_i = \rho(a) u_i \) for every \( a \in F_{d,K} \) and every \( i = 1, \ldots, n. \) It follows that \( \langle \psi_n(a) u, u \rangle = \langle \rho_n(a) u, u \rangle \) for every \( a \in F_d \) and every \( u \in (D_\psi)^n \), hence (C) is equivalent to
Proposition 3.6. Let \( (\rho, D_\rho) \) be a convex cone in real vector space \( V \). An element \( v \in V \) belongs to \( C^\circ \) if and only if for every element \( v \in V \) there exists \( n \in \mathbb{N} \) such that \( ne + v \in C \). If one of these equivalent conditions is true then for every \( v \in V \) the following are equivalent:

(D) \( \langle \rho_n(q)u, u \rangle \geq 0 \) for every finite-dimensional \( * \)-representation \((\rho, D_\rho)\) of \( F_d \) and every \( u \in (D_\rho)^n \) such that \( \langle \rho_n(p_i)u, u \rangle \geq 0 \) for all \( i = 1, \ldots, k \).

If \((\rho, D_\rho)\) is a finite-dimensional \( * \)-representation of \( F_d \) and \( C_1, \ldots, C_d \) are matrices that belong to the operators \( \rho(X_1), \ldots, \rho(X_d) \) in some orthonormal basis of \( D_\rho \), then \((\rho, D_\rho)\) is unitarily equivalent to the \( * \)-representation \((ev_C; \mathbb{R}^m)\), where \( C = (C_1, \ldots, C_d) \), and \((\rho_n, (D_\rho)^n)\) is unitarily equivalent to \((ev_C)_n, \mathbb{R}^{mn})\). Hence, (D) is equivalent to

(E) \( \langle (ev_C)_n(q)v, v \rangle \geq 0 \) for every \( m \in \mathbb{N} \), every \( C \in (M_m)^d \) and every \( v \in \mathbb{R}^{mn} \) such that \( \langle (ev_C)_n(p_i)v, v \rangle \geq 0 \) for all \( i \).

We claim that (E) is equivalent to (1). Let \( P \) be the matrix of the transpose mapping on \( M_{mn} \) in the basis \( E_{11}, \ldots, E_{1n}, \ldots, E_{n1}, \ldots, E_{nn} \). Note that \( P \) is a permutation (hence orthogonal) matrix such that \( B \otimes I_{mn} = P^T(I_{mn} \otimes B)P \) for every \( B \in M_{mn} \). In particular, we have

\[
\text{ev}_{mn \otimes C}(f) = ev_C(f) \otimes I_{mn} = P^T(I_{mn} \otimes ev_C(f))P = P^T(mn \otimes ev_C(f))P
\]

for every \( f \in M_n(F_d) \). Writing \( u = (u_1, \ldots, u_{mn}) \) and \( v = Pu = (v_1, \ldots, v_{mn}) \), we get that

\[
\langle (ev_{mn \otimes C})_n(q)u, u \rangle = \langle P^T(mn \otimes (ev_C)_n(q))Pu, u \rangle = \sum_{i=1}^{mn} v_i^T(ev_C)_n(q)v_i = \text{Tr}((ev_C)_n(q)B) \]

which belongs to \( \Sigma_{mn} \). Conversely, every element \( B \) of \( \Sigma_{mn} \) has a decomposition \( B = \sum_{i=1}^{mn} v_i v_i^T \) with \( v_i \in \mathbb{R}^{mn} \) and for \( u = P^T(v_1, \ldots, v_n) \) we have that \( \langle (ev_{mn \otimes C})_n(q)u, u \rangle = \text{Tr}((ev_C)_n(q)B) \).

A subset \( C \) of a real vector space \( V \) is a convex cone if \( C + C \subseteq C \) and \( \mathbb{R}^+ C \subseteq C \). Write \( \overline{C} \) for the closure and \( C^\circ \) for the interior of \( C \) in the finest locally convex topology of \( V \). Write \( C^\vee \) for the set of all real linear functionals \( f \) on \( V \) such that \( f(C) \geq 0 \). The following is well known and easy to prove.

**Proposition 3.6.** Let \( C \) be a convex cone in real vector space \( V \). An element \( e \in V \) belongs to \( C^\circ \) if and only if for every element \( v \in V \) there exists \( n \in \mathbb{N} \) such that \( ne + v \in C \). If one of these equivalent conditions is true then for every \( v \in V \) the following are equivalent:
(1) \( v \in \overline{C} \),
(2) \( v + \varepsilon e \in C \) for every real \( \varepsilon > 0 \), and
(3) \( f(v) \geq 0 \) for every \( f \in C' \).

Under the same assumptions, the following are equivalent:

(1') \( v \in C^\circ \),
(2') \( v - \varepsilon e \in C \) for some real \( \varepsilon > 0 \), and
(3') \( f(v) > 0 \) for every \( f \in C^\circ \).

The following is the archimedean version of Theorem 3.1.

Theorem 3.7. Pick \( n \) and \( d \). For every elements \( p_1, \ldots, p_k \in S_n(F_d) \) such that the cone \( C := \Sigma_n(F_d) + \mathbb{R}^+ p_1 + \ldots + \mathbb{R}^+ p_k \) is archimedean (i.e. \( 1 \in C^\circ \) the following are equivalent for every \( q \in S_n(F_d) \):

(a) \( \text{Tr}((ev_C)_n(q)B) \geq 0 \) for every \( m \in \mathbb{N} \), \( C \in (M_m)^d \) and \( B \in \Sigma_{mn} \) such that \( \text{Tr}((ev_C)_n(p_i)B) \geq 0 \) for all \( i = 1, \ldots, k \).
(b) \( q + \varepsilon I_n \in C \) for every real \( \varepsilon > 0 \).

Under the same assumptions, the following are equivalent:

(a') \( \text{Tr}((ev_C)_n(q)B) > 0 \) for every \( m \in \mathbb{N} \), \( C \in (M_m)^d \) and \( B \in \Sigma_{mn} \setminus \{0\} \) such that \( \text{Tr}((ev_C)_n(p_i)B) \geq 0 \) for all \( i = 1, \ldots, k \).
(b') \( q - \varepsilon I_n \in C \) for some real \( \varepsilon > 0 \).

Proof. By Theorem 3.1, (a) is equivalent to \( q \in \overline{C} \) and by Proposition 3.6 applied to \( V = S_n(F_d) \) and \( e = I_n \), \( q \in \overline{C} \) is equivalent to (b).

To prove the second part, consider the following claim:

(A') \( f(q) > 0 \) for every real linear functional \( f \) on \( S_n(F_d) \) such that \( f(\Sigma_n(F_d)) \geq 0 \) and \( f(p_1) \geq 0, \ldots, f(p_k) \geq 0 \).

We can prove that (A') is equivalent to (a') by following the proof of Theorem 3.1 (the equivalence of assertions (A) and (1)). Applying Proposition 3.6 as above, we see that (A') is equivalent to (b'). \( \square \)

4. General \(*\)-algebras

Let \( \mathcal{A} \) be a fixed \(*\)-algebra and \( \mathcal{R}_{\text{all}} = \mathcal{R}_{\text{all}}(\mathcal{A}) \) the class of all \(*\)-representations of \( \mathcal{A} \). In K. Schmüdgen’s approach to noncommutative real algebraic the elements of \( \text{Sym}(\mathcal{A}) = \{ a \in \mathcal{A} \mid a = a^* \} \) are considered as ‘noncommutative real polynomials’ and the elements of some fixed subclass \( \mathcal{R} \) of \( \mathcal{R}_{\text{all}} \) are considered as ‘noncommutative real points’. Interesting choices for \( \mathcal{R} \) include the class \( \mathcal{R}_{\text{fin}} \) of all finite-dimensional \(*\)-representations of \( \mathcal{A} \) and the class \( \mathcal{R}_{\text{bnd}} \) of all bounded \(*\)-representations of \( \mathcal{A} \). The positivity set of a given subset \( S \) of \( \text{Sym}(\mathcal{A}) \) is defined by

\[
K_S^{\mathcal{R},\text{sch}} := \{ (\pi, D_\pi) \in \mathcal{R} \mid \pi(s) \succeq 0 \text{ for all } s \in S \}
\]
and the problem is to compute the saturations
\[
\text{Sat}_{\geq}^{R, \text{sch}}(S) := \{ q \in \text{Sym}(A) \mid \pi(q) \geq 0 \text{ for all } (\pi, D_\pi) \in K_{S, \text{sch}}^R \}
\]
\[
\text{Sat}_{\leq}^{R, \text{sch}}(S) := \{ q \in \text{Sym}(A) \mid \pi(q) \leq 0 \text{ for all } (\pi, D_\pi) \in K_{S, \text{sch}}^R \}
\]
and
\[
\text{Sat}_{\geq}^{\text{sch}, R}(S) := \{ a \in \text{Sym}(A) \mid \pi(a) \geq 0 \text{ for all } (\pi, D_\pi) \in K_{S}^{\text{sch}, R} \}.
\]

**Example 4.1.** If \( A = M_n(\mathcal{P}_d), S \subseteq \text{Sym}(A) \) and \( R(A) \) is the set of all mappings \( \text{ev}_a : A \to M_n, \text{ev}_a(P(X)) = P(a) \), where \( a \in \mathbb{R}^d \), then
\[
\text{Sat}_{\geq}^{\text{sch}}(S) = \text{Sat}_{\geq}^{R(A), \text{sch}}(S).
\]
If \( B = M_n(F_d) \) and \( S \subseteq \text{Sym}(B) \) then
\[
\text{Sat}_{\geq}^{\text{sch}}(S) = \text{Sat}_{\geq}^{R(\text{bnd}(B)), \text{sch}}(S).
\]
The sets \( \text{Sat}_{\geq}^{\text{sch}}(S) \) and \( \text{Sat}_{\geq}^{\text{sch}}(S) \) were defined in the introduction.

Recall that a subset \( M \) of \( \text{Sym}(A) \) is called a quadratic module in \( A \) if \( M + M \subseteq M, 1 \in M \) and \( a^* M a \subseteq M \) for every \( a \in A \). Write \( \overline{M} \) for the closure and \( M^\circ \) for the interior of \( M \) in the finest locally convex topology of the real vector space \( \text{Sym}(A) \). A quadratic module \( M \) is archimedean if for every \( a \in \text{Sym}(A) \) there exists a real positive \( k \) such that \( k \cdot 1 + a \in M \). By Proposition \[3.6\] \( M \) is archimedean iff \( 1 \in M^\circ \).

**Theorem 4.2.** Let \( A \) be a \( * \)-algebra and \( S \subseteq \text{Sym}(A) \). Write \( M_S \) for the smallest quadratic module in \( A \) which contains \( S \). Then
\[
\text{Sat}_{\geq}^{R, \text{sch}}(S) = \overline{M_S}
\]
Moreover, if \( M_S \) is archimedean then
\[
\text{Sat}_{\geq}^{R(\text{bnd}), \text{sch}}(S) = \{ a \in \text{Sym}(A) \mid \varepsilon \cdot 1 + a \in M_S \text{ for some } \varepsilon \in \mathbb{R}^+ \} = (M_S)^\circ,
\]
\[
\text{Sat}_{\leq}^{R(\text{bnd}), \text{sch}}(S) = \{ a \in \text{Sym}(A) \mid \varepsilon \cdot 1 + a \in M_S \text{ for all } \varepsilon \in \mathbb{R}^+ \} = M_S,
\]
\[
\text{Sat}_{\geq}^{R(\text{bnd}), \text{sch}}(S) = \{ f \in \text{Sym}(A) \mid -1 \in M_{S \cup \{-f\}} \}.
\]

The proof of the first part is almost the same as the proof of (i) \( \iff \) (ii) in \[21\] Proposition 3 or the proof of (2) \( \iff \) (B) in our Theorem \[3.1\]. The second part is the same as \[3\] Theorem 12 and \[4\] Theorem 5, or Propositions 14-16 in \[21\]. Note that the theorem of Helton & McCullough follows from Theorem 4.2 and Example 4.1. For Scherer & Hol you also need an observation from the proof of \[14\] Theorem 13 that \( \text{Sat}_{\geq}^{R(A), \text{sch}}(S) = \text{Sat}_{\geq}^{R(\text{bnd}(A)), \text{sch}}(S) \) when \( M_S \) is archimedean.

Our main results do not fit into Schmüdgen’s approach but they fit instead into the approach that was outlined (in the case of free algebras) by Helton, McCullough and Putinar in \[12\]. In this approc
‘noncommutative real polynomials’ are the same as above, i.e. elements of Sym(A), but ‘noncommutative real points’ are different - they are triples (π, Dπ, v), where (π, Dπ) belongs to R and v belongs to Dπ \ {0}. Write pt(R) for the set of all such triples.

Let S be a subset of Sym(A). Its positivity set is defined by

\[ K_S^{R_{\text{hmp}}} := \{(\pi, D\pi, v) \in \text{pt}(R) \mid \langle \pi(s)v, v \rangle \geq 0 \text{ for all } s \in S\} \]

and the corresponding saturations are defined by

\[ \text{Sat}_{R_{\text{hmp}}}^>(S) := \{q \in \text{Sym}(A) \mid \langle \pi(q)v, v \rangle > 0 \text{ for all } (\pi, D\pi, v) \in K_S^{R_{\text{hmp}}}\} \]

and

\[ \text{Sat}_{R_{\text{hmp}}}^\geq(S) := \{q \in \text{Sym}(A) \mid \langle \pi(q)v, v \rangle \geq 0 \text{ for all } (\pi, D\pi, v) \in K_S^{R_{\text{hmp}}}\}. \]

**Example 4.3.** Recall the definition of the set Sat′_>(S) from the introduction. If A = M_n(P), S ⊆ Sym(A) and R′(A) = \{n ⊕ ev_a \mid a ∈ R^d\} where (n ⊕ ev_a)(P) = n ⊕ P(a) for every P = P(X) ∈ A, then

\[ \text{Sat}_>(S) = \text{Sat}_{R_{\text{hmp}}}^{R'(A)}(S). \]

This follows from the identity \(\langle (n \oplus ev_a)(P)(v_1, \ldots, v_n), (v_1, \ldots, v_n) \rangle = \sum_{i=1}^n v_i^T ev_a(P)v_i = \text{Tr}(ev_a(P)B)\) where \(B = \sum_{i=1}^n v_i v_i^T \in \Sigma_n\). (Compare with equivalence (F) ⇔ (1) in the proof of Theorem 3.1)

We would like to point out that in general

\[ \text{Sat}_{R_{\text{hmp}}}^{R(A)}(S) \supseteq \text{Sat}_{R_{\text{hmp}}}^{R'(A)}(S) \supseteq \text{Sat}_{R_{\text{hmp}}}^{R_{\text{fin}}(A)}(S) \]

Both inclusions are clear (because larger representation class means larger positivity set and so smaller saturation); we just have to prove that they are proper.

(a) For the following constant elements of S_2(P_d)

\[ p_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, p_2 = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \]

we have that

\[ Q \in \text{Sat}_{R_{\text{hmp}}}^{R_{\text{fin}}(A)}(\{p_1, p_2\}) \setminus \text{Sat}_{R_{\text{hmp}}}^{R_{\text{hmp}}(A)}(\{p_1, p_2\}). \]

**Proof.** We claim that for every \(v = [x, y]^T ∈ \mathbb{R}^2\) such that \(v^*p_1v ≥ 0\) and \(v^*p_2v ≥ 0\), we also have that \(v^*Qv ≥ 0\). The claim is clearly true if \(x = 0\). If \(x ≠ 0\), then we can assume that \(x = 1\). We get \(v^*p_1v ≥ 0\) iff \(y^2 ≤ 1\), \(v^*p_2v ≥ 0\) iff \((y - 1)^2 ≥ 1\) and \(v^*Qv ≥ 0\) iff \((y + 1)^2 ≤ 1\), which implies the claim. On the other hand, note that \(e_1^TP_1e_1 + e_2^TP_1e_2 ≥ 0\) and \(e_1^TP_2e_1 + e_2^TP_2e_2 ≥ 0\) but \(e_1^TQe_1 + e_2^TQe_2 < 0\) where \(e_1, e_2\) is the standard basis of \(\mathbb{R}^2\).
(b) For \( p = X_1 \) and \( q = X_3 \) from \( \mathcal{A} = \mathbb{R}[X_1] \) we have that
\[
q \in \text{Sat}_{\geq}^{\mathcal{R}(\mathcal{A}), \text{hmp}}(\{p\}) \setminus \text{Sat}_{\geq}^{\mathcal{R}_{\text{fin}}(\mathcal{A}), \text{hmp}}(\{p\}).
\]

Proof. Since \( a_1 \geq 0 \) implies that \( a_1^3 \geq 0 \) for every \( a_1 \in \mathbb{R} \), it follows that \( q \in \text{Sat}_{\geq}^{\mathcal{R}(\mathcal{A}), \text{hmp}}(\{p\}) \). On the other hand, for \( \pi: u(X) \mapsto [u(-2)] \oplus [u(1)] \) and \( v = [1, 2] \), we have that \( \langle \pi(p(X))v, v \rangle = p(-2)1^2 + p(1)2^2 = 2 \geq 0 \) and \( \langle \pi(q(X))v, v \rangle = q(-2)1^2 + q(1)2^2 = -4 < 0 \), hence \( q \not\in \text{Sat}_{\geq}^{\mathcal{R}_{\text{fin}}(\mathcal{A}), \text{hmp}}(\{p\}) \).

Example 4.4. Recall the definition of the set \( \text{Sat}''(S) \) from the introduction. If \( B = M_n(\mathbb{F}) \), \( S \subseteq \text{Sym}(B) \) and \( \mathcal{R}(B) \) is the set of all mappings \( (ev_C)_n \) where \( C \) is a \( d \)-tuple of same size matrices then
\[
\text{Sat}''(S) = \text{Sat}_{\geq}^{\mathcal{R}(B), \text{hmp}}(S).
\]

By the proof of Theorem 3.1 we also have that
\[
\text{Sat}_{\geq}^{\mathcal{R}'(B), \text{hmp}}(S) = \text{Sat}_{\geq}^{\mathcal{R}_{\text{fin}}, \text{hmp}}(S) = \text{Sat}_{\geq}^{\mathcal{R}_{\text{all}}(B), \text{hmp}}(S).
\]

A subset \( N \) of \( \text{Sym}(\mathcal{A}) \) will be called a quadratic cone in \( \mathcal{A} \) if \( N + N \subseteq N \), \( \mathbb{R}^{\geq} \cdot N \subseteq N \) and \( a^*a \in N \) for every \( a \in \mathcal{A} \). The following general result can be extracted from the proofs of Theorems 3.1 and 3.7. It is an analogue of Theorem 4.2.

Theorem 4.5. Let \( \mathcal{A} \) be a \( \ast \)-algebra and \( S \subseteq \text{Sym}(\mathcal{A}) \). Write \( N_S \) for the smallest quadratic cone in \( \mathcal{A} \) which contains \( S \). Then
\[
\text{Sat}^{\mathcal{R}_{\text{all}}, \text{hmp}}(S) = N_S.
\]

Moreover, if \( N_S \) is archimedean then
\[
\text{Sat}^{\mathcal{R}_{\text{bnd}}, \text{hmp}}(S) = \{ a \in \text{Sym}(\mathcal{A}) \mid \epsilon \cdot 1 + a \in N_S \text{ for all } \epsilon \in \mathbb{R}^{\geq} \} = N_S
\]
and
\[
\text{Sat}^{\mathcal{R}_{\text{bnd}}, \text{hmp}}(S) = \{ a \in \text{Sym}(\mathcal{A}) \mid \epsilon \cdot 1 + a \in N_S \text{ for some } \epsilon \in \mathbb{R}^{\geq} \} = (N_S)^{\circ}.
\]

For \( \mathcal{A} = M_n(\mathbb{F}) \), Theorem 4.5 extends Theorem 3.1. However, if \( \mathcal{A} = M_n(\mathbb{P}) \), it extends neither Theorem 2.1 nor the solution of (the usual or the matrix) Hilbert’s 17th problem. The matrix version of the Hilbert’s 17th problem was solved independently by [6] and [17]. For a constructive proof see Proposition 10 in [20] which precedes [13].

For \( S = \emptyset \), Theorem 4.5 is the same as Theorem 4.2. The case \( N_\emptyset \) archimedean (i.e. \( \mathcal{A} \) algebraically bounded) is known as Vidav-Handelman theory, see [8, Section 1] and [22].
References


