A SHORT NOTE ON THE HIGHER LEVEL VERSION OF THE KRULL-BAER THEOREM

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Abstract. In [2] Klep and Velišek generalized the Krull–Baer theorem for higher level preorderings to the non-commutative setting. A \( n \)-real valuation \( v \) on a skew field \( D \) induces a group homomorphism \( \pi \). A section of \( \pi \) is a crucial ingredient of the construction of a complete preordering on the base field \( D \) such that its projection on the residue skew field \( k_v \) equals the given level \( 1 \) ordering on \( k_v \). In the article we give a proof of the existence of the section of \( \pi \), which was left as an open problem in [2], and thus complete the generalization of the Krull–Baer theorem for preorderings.

1. Introduction

In the breakthrough article on orderings of higher level [1] Becker proved a higher level version of the Krull-Baer theorem regarding complete preorderings [1, Satz 2.4] which explicitly describes all complete preorderings which are lifts of a given level \( 1 \) ordering on the residue field. One of the crucial ingredients of the construction of preorderings is a section of a surjective homomorphism \( \overline{\pi} : D^\times / \prod_n D^\times \to \Gamma / \prod_n \Gamma \) induced by a valuation \( v : D \to \Gamma \cup \{\infty\} \). By introducing the notion of \( n \)-real valuations Klep and Velišek [2] generalized Becker’s version of the Krull-Baer theorem to skew fields [2, Theorem 6] modulo the existence of the section of \( \overline{\pi} \), which was left as an open problem.

In the following sections we use the theory of abelian groups with finite exponent to construct a section of the homomorphism \( \overline{\pi} \) which together with [2, Theorem 6] gives us a generalization of Becker’s result for skew fields.

2. Preliminaries

In this section we give all information about valuations, orderings and complete preorderings of skew fields which is needed in this paper. We restrict to the definitions and theorems which are actually used and presuppose basic knowledge of commutative valuations as well as commutative orderings. Throughout this paper \( D \) will denote a skew field.

Let \( G \) be an arbitrary group. Throughout the paper \( \prod_n G \) will denote the subgroup of \( G \) generated by \( n \)-th powers and multiplicative commutators. The elements of \( \prod_n G \) are called the \textit{permuted products} of \( n \)-th powers of elements of \( G \). It is clear that every element of \( \prod_n G \) can be written as a product of terms

\[ a_1 \cdots a_n \]

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which are a result of applying a permutation on a certain product of \( n \)-th powers and vice versa. For example: \( x^2y^3yz^2y^3z \) is a permuted product of 4-th powers.

**Definition:** Let \( n \in \mathbb{N} \). A subset \( P \subseteq D \) is called an ordering of level \( n/2 \) (of exponent \( n \)) if
\[
\prod_n D^x \subseteq P, \quad P + P \subseteq P, \quad P \cdot P \subseteq P
\]
and \( D^x/P^x \) is a cyclic group.

**Definition:** A set \( P \subseteq D \) is a complete preordering of exponent \( n \) or level \( n/2 \) if
\[
\prod_n D^x \subseteq P, \quad -1 \notin P, \quad P + P \subseteq P, \quad P \cdot P \subseteq P, \quad a^2 \in P \Rightarrow a \in P \cup -P.
\]

Note that an ordering of level \( n \) is also a complete preordering of level \( n \) and every complete preordering of level \( n \) can be extended to an ordering of level \( n \) (see e.g. [5, Theorem 3.13]).

All valuations considered will be invariant. A valuation with (not necessarily commutative) value group \( \Gamma_v \) will be denoted by \( v: D \rightarrow \Gamma_v \cup \{\infty\} \) and \( \mathcal{O}_v, m_v, k_v, \Gamma_v \) will represent its valuation ring, its maximal ideal, its residue skew field and its value group respectively (see Schilling [4] for more details). For \( a \in \mathcal{O}_v, \overline{a} := a + m_v \) will denote its image in \( k_v \).

**Definition:** A valuation \( v \) is said to be compatible with a complete preordering \( P \) if \( 1 + m_v \subseteq P \).

See e.g. [5] for more details.

**Definition:** The valuation \( v \) is \( n \)-real if \( k_v \) admits an ordering \( \mathcal{P} \) of level 1 with the property \( \mathcal{O}_v^x \cap \prod_{2n} D^x \subseteq \mathcal{P} \). An ordering \( \mathcal{P} \) of \( k_v \) will be called \( n \)-compatible with \( v \) if \( \mathcal{O}_v^x \cap \prod_{2n} D^x \subseteq \mathcal{P} \).

For more results on \( n \)-real valuations see [2].

### 3. Lifting preorderings

The main ingredient of the proof of [1] Satz 2.4 and the proof of [2] Theorem 6 is a section of a surjective homomorphism \( \overline{v}: D^x/\prod_n D^x \rightarrow \Gamma/\prod_n \Gamma \) which is induced by a valuation \( v: D \rightarrow \Gamma \cup \{\infty\} \). Its existence in the noncommutative case was left as an open problem in [2]. In the following section we present a construction of a section of the homomorphism \( \overline{v} \) which enables us to generalize the theorem [1, Satz 2.4.] to the noncommutative setting.

**Lemma 1:** Let \( v: D \rightarrow \Gamma \cup \{\infty\} \) be a valuation on \( D \). Then \( v \) induces a surjective homomorphism \( \overline{v}: D^x/\prod_n D^x \rightarrow \Gamma/\prod_n \Gamma \) with a section \( \mu: \Gamma/\prod_n \Gamma \rightarrow D^x/\prod_n D^x \), i.e. \( \overline{v} \circ \mu = \text{id} \).

**Proof.** The existence of \( \overline{v} \) is clear. For an arbitrary group \( G \) let us denote by \( [G, G] \) the commutator subgroup of \( G \). Note that \( [G, G] \) is a normal subgroup of \( G \) and \( \prod_m G \) for all \( m \in \mathbb{N} \). Let us denote \( \Gamma^{ab} = \Gamma/[\Gamma, \Gamma] \). Hence, by isomorphism
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g_{n} \Gamma \cong \Gamma \Gamma_{ab}^{n}

Without loss of generality we can take the canonical isomorphism to be the identity and let \( n = \prod_{p} p^{\alpha_{p}} \) be the prime decomposition of \( n \). By [3, Chapter 8 Exercise 1] we have

\[
\frac{\Gamma_{ab}}{\Gamma_{ab}} = \bigoplus_{p} \frac{\Gamma_{ab}}{\Gamma_{ab}^{p}}
\]

Thus, it is enough to find \( \mu_{p} : \Gamma_{ab} / (p^{\alpha_{p}} \Gamma_{ab}) \to D^{\times} / \prod_{n} D^{\times} \) such that \( \overline{\varphi} \circ \mu_{p} = \text{id} \bigg|_{\frac{\Gamma_{ab}}{p^{\alpha_{p}} \Gamma_{ab}}} \) for every prime \( p \) from the prime decomposition of \( n \).

The group \( \Gamma_{ab} / (p^{\alpha_{p}} \Gamma_{ab}) \) is a bounded \( p \)-group, thus by [3, Theorem 17.2]

\[
\frac{\Gamma_{ab}}{p^{\alpha_{p}} \Gamma_{ab}} = \bigoplus_{i \in I} (a_{i})
\]

for some \( a_{i} \in \Gamma_{ab} / (p^{\alpha_{p}} \Gamma_{ab}) \). Since \( \overline{\varphi} \) is surjective, we can pick a \( b_{i} \in D^{\times} / \prod_{n} D^{\times} \) such that \( \overline{\varphi}(b_{i}) = a_{i} \) for every \( i \in I \). Take an arbitrary \( \sum_{i \in I} k_{i} a_{i} \in \Gamma_{ab} / (p^{\alpha_{p}} \Gamma_{ab}) \), \( k_{i} \in \mathbb{Z} \) and define

\[
\mu_{p} \left( \sum_{i \in I} k_{i} a_{i} \right) = \prod_{i \in I} b_{i}^{k_{i}}.
\]

Clearly \( \mu_{p} \) is a homomorphism and \( \overline{\varphi} \circ \mu_{p} = \text{id} \bigg|_{\frac{\Gamma_{ab}}{p^{\alpha_{p}} \Gamma_{ab}}} \) which finishes the proof.

Using the Lemma [1] and [2] Theorem 6 we get the constructive version of the Baer-Krull like theorem for complete preorderings which generalizes the commutative theorem [1, Satz 2.4].

**Corollary 2** (c.f. Satz 2.4 in [1]): Assume \( n \in \mathbb{N} \) is even, \( v : D \to \Gamma \cup \{\infty\} \) is \( c_{2} \)-real and let \( \overline{\varphi} : D^{\times} / \prod_{n} D^{\times} \to \Gamma / \prod_{n} \Gamma \) be the induced group homomorphism. Denote by \( \mu : \Gamma / \prod_{n} \Gamma \longrightarrow D^{\times} / \prod_{n} D^{\times} \) a section of \( \overline{\varphi} \) as in Lemma [1]. Fix a set of representatives \( \Omega \subseteq D^{\times} \) for \( \mu(\Gamma / \prod_{n} \Gamma) \) with \( 1 \in \Omega \).

Let \( P \) be a level 1 ordering of \( k_{v} \) that is \( n \)-compatible with \( v \). Suppose \( \Gamma_{0} \) is a subgroup of \( \Gamma \) containing \( \prod_{n} \Gamma \) such that the Sylow 2-subgroup of \( \Gamma / \Gamma_{0} \) is cyclic of order \( 2^{r} \), \( r \geq 0 \) and \( \chi : \Gamma_{0} \longrightarrow k_{v}^{\times} / P^{\times} \) is a character satisfying \( \chi(\prod_{n} \Gamma) = 1 \) and \( \chi(\Gamma_{0} \cap \prod_{n} \Gamma) \neq 1 \) if \( r \geq 1 \).

Define \( \mathcal{U}_{0} \subseteq \Omega \) to be the system of representatives of \( \mu(\Gamma_{0} / \prod_{n} \Gamma) \) and for every \( a \in \mathcal{U}_{0} \) denote \( M_{a} := \{ \varepsilon \in \mathcal{O}_{\Omega}^{\times} | \chi(v(a)) = \varepsilon \overline{\varphi} \} \). Then:

(a) \( P := \bigcup_{a \in \mathcal{U}_{0}} a M_{a} \prod_{n} D^{\times} \) is a complete preordering of exponent \( n \) compatible with \( v \) and induces the given ordering \( \overline{\varphi} \) of \( k_{v} \).

(b) Every complete preordering of exponent \( n \) compatible with \( v \) that induces the ordering \( \overline{\varphi} \) of \( k_{v} \) is obtained in this way.
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REFERENCES


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