A VARIANT OF HIGHER PRODUCT LEVELS OF INTEGRAL DOMAINS AND ∗-DOMAINS

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Abstract. For every domain $R$ and every even integer $n$ we define $ms_n(R)$ (resp. $ps_n(R)$) as the smallest number $k$ such that 0 is a sum of $k + 1$ products (resp. permuted products) of $n$-th powers of nonzero elements from $R$. There are many results about $ps_n$ in the literature but nothing is known about $ms_n$.

We prove two results about $ms_n$ of twisted Laurent series rings $R((x, ω))$. The first result is that if $ms_2(R) = ∞$ and $ω$ has order $n/2$ in $Aut(R)$, then $ms_n(R((x, ω))) = ∞$. The second result is that there exist $R$ and $ω$ such that $ms_n(R((x, ω))) = ∞$ and $ps_n(R((x, ω))) < ∞$. (Take $k = \frac{n}{2} - 1$, $R = \mathbb{R}(t_1, \ldots, t_k)$ and $ω(f(t_1, \ldots, t_k)) = f(-t_k, t_1 - t_k, \ldots, t_{k-1} - t_k).$)

Finally, we define $ms_n$ and $ps_n$ of a domain $R$ with involution. For a certain involution on $R((x, ω))$, we prove analogues of the first and the second result.

1. Introduction

The $n$-th level of a field $F$, denoted by $s_n(F)$, is the length of the shortest representation of $−1$ as a sum of $n$-th powers of elements from $F$. The study of $s_n(F)$ was initiated by J.-P. Joly in [9] and continued by E. Becker in a series of papers (e.g. [2], [3]). The case $n = 2$ is classical, see [14].

The results about $s_n(F)$ do not carry over to skew-fields because of the fact that the product of two $n$-th powers need not be an $n$-th power. Therefore, for a skew-field $D$ and number $n$, we have besides $s_n(D)$ at least two other related invariants, $ms_n(D)$ and $ps_n(D)$, which are defined as the number of terms in the shortest representation of $−1$ as a sum of products (in the case of $ms_n$) or permuted products (in the case of $ps_n$) of $n$-th powers of elements from $D$. The invariant $ps_n(D)$ for $n > 2$ was introduced in [4] and studied later in [10] and [6]. The motivation for studying $ps_n$ comes from the theory of higher level orderings. Namely, skew-fields with $ps_n(D) = 1$ are furthest away from having an ordering of level $n/2$ while skew-fields with $ps_n(D) = ∞$ always have such an ordering, see [7] and [15]. Our motivation for

Date: 9. 5. 2004.
studying $ms_n$ will be explained in a separate paper where we will define
a variant of higher level orderings. It measures how close is $D$ to having
this type of ordering. For $n > 2$, the invariants $s_n(D)$ and $ms_n(D)$ have
not yet been studied. If $n = 2$ then $ps_n(D) = ms_2(D)$ because every
multiplicative commutator is a product of three squares, the standard
notation in this case is $s_π(D)$. Important references are [17], [18] and
[12] for $s_π(D) < \infty$ and [16], [1] for $s_π(D) = \infty$. Our $s_2(D)$ is usually
denoted by $s(D)$. It has been studied by many authors mostly for
quaternion algebras, the results are summarized in [13].

The next step is to pass from skew fields to integral domains (=as-
sociative rings without zero divisors). Let $R_n$ be the set of
$n$-th powers
of elements from $R$, $P_n(R)$ the set of products of elements from $R_n$
and $Π_n(R)$ for the set of all permutations of products from $P_n(R)$.
For example, if $x, y, z \in R$, then $(xyz)(zxy)^3 \in Π_4(R)$, since it is a
permutation of $x^4y^4z^4 \in P_4(R)$. We define
\[
s_n(R) = \min\{k \mid \exists z_0, \ldots , z_k \in R_n \setminus \{0\} : 0 = \sum_{i=0}^k z_i\},
ms_n(R) = \min\{k \mid \exists z_0, \ldots , z_k \in P_n(R) \setminus \{0\} : 0 = \sum_{i=0}^k z_i\},
ps_n(R) = \min\{k \mid \exists z_0, \ldots , z_k \in Π_n(R) \setminus \{0\} : 0 = \sum_{i=0}^k z_i\},
\]
where $\min \emptyset = \infty$. Obviously, we have $ps_n(R) \leq ms_n(R) \leq s_n(R)$ for
every $n$ and $R$. If $n$ is odd then all are equal to 1. This definition of $ms_n$
and $ps_n$ extends the definition from the skew-field case, however $s_n(D)$
need not be equal to $s_n(D)$. The number $s_2(D)$ is usually denoted by
$s(D)$ and called sublevel, see [13].

We will prove two results about $ms_n$ for twisted Laurent series rings:

**Theorem.** If $ms_2(R) = \infty$ and $ω$ has order $m$ in $\text{Aut}(R)$, then
\[ms_{2m}(R((x, ω))) = \infty.\]

**Theorem.** For every $m \geq 2$, there exist $R$ and $ω$ such that
\[ms_{2m}(R((x, ω))) = \infty \text{ and } ps_{2m}(R((x, ω))) < \infty.\]

Let $(D, *)$ be a skew-field with involution. An element of the form
$xx^*$, $x \in D$, is called a hermitian square in $D$. For every natural num-
ber $m$ we define the $2m$-th hermitian level $s_{2m}(D, *)$ as the number of
terms in the shortest representation of $-1$ as a sum of $m$-th powers
of hermitian squares. The results about $s_2(D, *)$ are surveyed in [13].
Again, the product of two hermitian squares need not be a hermitian
square. Therefore, we have at least two related invariants, $ms_{2m}(D, *)$
and \(ps'_2m(D, \ast)\), which are defined as the number of terms in the shortest representation of \(-1\) as a sum of products (resp. permuted products) of \(m\)-th powers of hermitian squares. We do not break up hermitian squares when we permute them. Note that \(ms_4(D, \ast) = ps'_4(D, \ast)\). The invariants \(ms_{2m}(D, \ast)\) and \(ps'_{2m}(D, \ast)\) have not yet been studied.

The theory of \(\ast\)-orderings of higher level (see [5] which is motivated by [8]) suggests that another invariant, \(ps_{2m}(D, \ast)\), may be more natural than \(ps'_{2m}(D, \ast)\). Here, the set of nonzero permuted products of \(m\)-th powers of hermitian squares is defined as the subgroup of \(D \setminus \{0\}\) generated by all elements of the form \((dd')^m\) and \(dsd^{-1}s^{-1}\), where \(d, s \in D \setminus \{0\}\) and \(s = s^\ast\). If we require also that \(s\) is a hermitian square, we get \(ps'_{2m}(D, \ast)\). Hence, \(ps_{2m}(D, \ast) \leq ps'_{2m}(D, \ast)\). The invariant \(ps_{2m}(D, \ast)\) will not be considered in the sequel.

For every integer \(m\) and every domain with involution \((R, \ast)\) write \(H_{2m}(R, \ast)\) for the set of all \(m\)-th powers of hermitian squares in \(R\), \(P_{2m}(R, \ast)\) for the set of all products of elements from \(H_{2m}(R, \ast)\) and \(\Pi_{2m}(R, \ast)\) for the set of permuted products of elements from \(H_{2m}(R, \ast)\) (hermitian squares do not break up when permuted). We define

\[
\begin{align*}
{s'}_{2m}(R, \ast) &= \min\{k | \exists z_0, \ldots, z_k \in H_{2m}(R, \ast) \setminus \{0\} : 0 = \sum_{i=0}^{k} z_i\}, \\
ms_{2m}(R, \ast) &= \min\{k | \exists z_0, \ldots, z_k \in P_{2m}(R, \ast) \setminus \{0\} : 0 = \sum_{i=0}^{k} z_i\}, \\
ps'_{2m}(R, \ast) &= \min\{k | \exists z_0, \ldots, z_k \in \Pi_{2m}(R, \ast) \setminus \{0\} : 0 = \sum_{i=0}^{k} z_i\},
\end{align*}
\]

where \(\min \emptyset = \infty\). We have \(ps'_{2m}(R, \ast) \leq ms_{2m}(R, \ast) \leq s_{2m}(R, \ast)\). If \(R\) is a skew-field, then \(s_2(R, \ast) = s_{2}(R, \ast)\) and the definitions of \(ms_{2m}(R, \ast)\) and \(ps'_{2m}(R, \ast)\) are compatible with above.

If \(\omega\) is an automorphism of a domain \(R\) with involution \(*\) such that \(\omega(r)^\ast = \omega^{-1}(r^\ast)\) for every \(r \in R\), then \((\sum a_ix^i)^\ast = \sum \omega^i(a_i^\ast)x^i\) defines an involution on the twisted Laurent series ring \(R((x, \omega))\). We will prove two results about \(ms_n(R((x, \omega)), \ast)\).

**Theorem.** Let \(R\) be a \(*\)-domain with \(ms_2(R, \ast) = \infty\) and \(\omega\) an automorphism of \(R\) compatible with \(*\). If \(\omega\) has order \(m\) in \(\text{Aut}(R)\) then

\[ms_{2m}(R((x, \omega)), \ast) = \infty.\]

**Theorem.** For every \(m \geq 3\), there exist a \(*\)-domain \(R\) and an automorphism \(\omega\) of \(R\) compatible with \(*\) such that

\[ms_{2m}(R((x, \omega)), \ast) = \infty\] and \(ps'_{2m}(R((x, \omega)), \ast) < \infty\).

2. The Mal’cev Neumann construction

If \(R\) is a ring, \(\Gamma\) an ordered group and \(\omega : \Gamma \to \text{Aut}(R)\) an antihomomorphism of groups, then the Mal’cev Neumann ring \(R((\Gamma, \omega))\) consists
of all formal expressions \( a = \sum a_\gamma \gamma \), where \( \text{supp}(a) \) is a well-ordered subset of \( \Gamma \). Addition is defined termwise and multiplication by

\[
(a_\gamma \gamma)(b_\delta \delta) = a_\gamma \omega_\gamma(b_\delta) \gamma \delta.
\]

If \( R \) is a skew-field, then \( R((\Gamma, \omega)) \) is also a skew-field by [11], Lemma 14.17. If \( R \) is an integral domain, then \( R((\Gamma, \omega)) \) is clearly also an integral domain. If \( \Gamma = \mathbb{Z} \), then \( \omega \) is uniquely determined by \( \omega_1 \) and \( R((\Gamma, \omega)) \) is just the usual twisted Laurent series ring \( R((x, \omega_1)) \).

**Theorem 1.** Let \( A = R((\Gamma, \omega)) \), where \( R, \Gamma, \omega \) are as above. If \( R \) is an integral domain with \( \text{ms}_2(R) = \infty \) and \( n \) is an even number such that \( \omega_2 \) has order \( n/2 \) in \( \text{Aut}(R) \) for every \( \gamma \in \Gamma \), then \( \text{ms}_n(A) = \infty \).

**Proof.** If \( a = c_\gamma \), then \( a^n = (c_\gamma)^n = c_{\omega_1(n)} \omega_1(r) \cdots \omega_1(n-1)(r) \gamma^n \). Since \( \omega_{\frac{n}{2}+i}(c) = \omega_i(r) \) for every \( i = 0, \ldots, \frac{n}{2} - 1 \), it follows that \( a^n = r^{2\gamma} \gamma^n \), where \( r = c_{\omega_{\frac{n}{2}}(c)} \cdots \omega_{\frac{n}{2}+1}(c) \).

It follows that \( (c_\gamma_1)^n \cdots (c_k \gamma_k)^n = (r_1^{2\gamma_1} \gamma_1^n \cdots (r_k^{2\gamma_k} \gamma_k^n = r_1^{2\gamma_1 \cdots \gamma_k^n } \). The last equality follows from the fact that \( (r_\gamma^k)(s_\delta^n) = r_\omega \gamma_\delta \gamma^n \delta^n = rs_\gamma^\delta \) by induction on \( n \).

If \( \text{ms}_n(A) < \infty \), then there exist nonzero \( a_{i,j} \in A \) such that \( 0 = \sum_{i=1}^{m} a_{i,1} \cdots a_{i,k_i} \). If \( a_{i,j} = c_{i,j} \gamma_{i+j} + \text{higher terms} \), then

\[
a_{i,1} \cdots a_{i,k_i} = r_1^{2\gamma_1 \cdots r_k^{2\gamma_k} \gamma_1^n \cdots \gamma_k^n + \text{higher terms}.
\]

Let \( \gamma = \min\{ \gamma_{i,1} \cdots \gamma_{i,k_i} : i = 1, \ldots, m \} \) and let \( i_t, t = 1, \ldots, l \) be all indices for which \( \gamma_{i_t,1} \cdots \gamma_{i_t,k_t} = \gamma \). Then

\[
0 = \sum_{t=1}^{l} a_{i_t,1} \cdots a_{i_t,k_t} = (\sum_{t=1}^{l} \prod_{j=1}^{k_t} r_{i_t,j}^2) \gamma + \text{higher terms}
\]

By comparing the coefficients on both sides, we see that

\[
0 = \sum_{t=1}^{l} \prod_{j=1}^{k_t} r_{i_t,j}^2, \quad \text{where } r_{i_t,j} \neq 0,
\]

contradicting the assumption on \( R \). \( \square \)

When can we extend an involution from \( R \) to \( R((\Gamma, \omega)) \)? We say that \( * \) and \( \omega \) are compatible if \( \omega_\gamma(r)^* = \omega_{\gamma^{-1}}(r^*) \) for every \( r \in R \) and every \( \gamma \in \Gamma \). If \( * \) and \( \omega \) are compatible and \( \Gamma \) is abelian then

\[
(\sum a_\gamma \gamma)^* = \sum \omega_\gamma(a^*_\gamma) \gamma
\]

defines an involution on \( R((\Gamma, \omega)) \). Note that only relations \( (c_\gamma)^{**} = c_\gamma \) and \( ((c_\gamma)(c_\delta))^{**} = (c_\delta)^* (c_\gamma)^* \) have to be verified and that these are true by a straightforward computation.
Theorem 2. Let $A = R((\Gamma, \omega))$, where $R$ is a $*$-domain and $*$ is compatible with $\omega$. If $ms_2(R, *) = \infty$ and $m$ is a positive integer such that $\omega_\gamma$ has order $m$ in $\text{Aut}(R)$ for every $\gamma \in \Gamma$, then $ms_{2m}(A, *) = \infty$.

Proof. If $a = (c\gamma)(c\gamma)^* = c\omega_{\gamma^2}(c^*)\gamma^2$, then
\[
a^m = c\omega_{\gamma^2}(c^*)\omega_{\gamma^2}(c)\omega_{\gamma^4}(c^*) \cdots \omega_{\gamma^{2m-2}}(c)\omega_{\gamma^{2m}}(c^*)\gamma^{2m} = 
\]
where
\[
r = \begin{cases} 
   c\omega_{\gamma^2}(c^*) \cdots \omega_{\gamma^{m-1}}(c^*) & m \text{ odd} \\
   c\omega_{\gamma^2}(c^*) \cdots \omega_{\gamma^{m-2}}(c^*)c^* & m \text{ even}
\end{cases}
\]
In both cases we used that
\[
\omega_{\gamma^{2m-2k}}(c^*) = \omega_{\gamma-2k}(c^*) = \omega_{\gamma 2k}((c^*)^*) = \omega_{\gamma 2k}(c^*)
\]
for every $k = 0, \ldots, m$. In the second case we also used $\omega_{\gamma m}(c^*) = c^*$. The rest of the proof is the same as in Theorem 1. For example, we have that $\prod_{i=1}^k((c_i \gamma_i)(c_i \gamma_i)^*)^m = \prod_{i=1}^k(r_i r_i^* \gamma_i^{2m}) = \prod_{i=1}^k r_i r_i^* \prod \gamma_i^{2m}$. $\square$

3. Skew-fields with finite $ps_n$ and infinite $ms_n$

Let $m$ be a positive integer, $k = m - 1$ and $n = 2m$. Write $D_n = R((x, \omega))$, where $R = \mathbb{R}(t_1, \ldots, t_k)$ is the field of real rational functions in $k$ variables and $\omega$ is an automorphism of $R$ defined by $\omega(f(t_1, \ldots, t_k)) = f(-t_k, t_1 - t_k, \ldots, t_{k-1} - t_k)$. Since $R$ is a field, $D_n$ is a skew field by Lemma 14.17 in [11].

Theorem 3. With the notation above, $ps_n(D_n) \leq k$ and $ms_n(D_n) = \infty$.

Proof. To prove that $ms_n(D_n) = \infty$ we must check the assumptions of Theorem 1. The fact that $m_2(\mathbb{R}) = \infty$ implies that $m_2(R) = \infty$. Define an $m \times m$ matrix
\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & -1 \\
1 & 0 & \cdots & 0 & -1 \\
0 & 1 & \cdots & 0 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -1
\end{bmatrix}^T.
\]
A simple computation shows that $I + A + \ldots + A^{m-1} = 0$. Since $\omega(f(t_1, \ldots, t_k)) = f((t_1, \ldots, t_k)A^t)$, and $A^m = I$, we have that $\omega^m$ is the identity.
Write \( x^i \) for the image of \( i \in \Gamma \) in \( D_n \) and check that \([cx^i, dx^j] = (cx^i)(dx^j)(cx^i)^{-1}(dx^j)^{-1} = \omega(x)^{i,j} \). It follows that
\[
\sum_{i=1}^{k} [x^i, t_k] = \sum_{i=1}^{k} [1 \cdot x^i, t_k \cdot x^0] = \sum_{i=1}^{k} \frac{\omega(t_k)}{t_k} = \sum_{i=1}^{k} \frac{p_{x}(t_1, \ldots, t_k) A_{i}}{t_k} = \frac{-t_k}{t_k} = -1,
\]
so that \( p_{n}(D_n) \leq k \). □

Let \( m \geq 3 \) be a positive integer, \( n = 2m \) and \( \xi \) a primitive \( m \)-th root of 1. Write \( K_n = R((x, \omega)) \), where \( R = \mathbb{C}(z) \) and \( \omega(f(z)) = f(\xi z) \). The field \( R \) has a natural involution given by
\[
\left( a_0 z^k + \ldots + a_k \right)^* = \frac{\bar{a}_0 z^k + \ldots + \bar{a}_k}{b_0 z^l + \ldots + b_l}.
\]
This involution is clearly compatible with \( \omega \), hence it extends to an involution on \( K_n \) by \( (\sum f_i(z)x^i)^* = \sum f_i^*(\xi^i z)x^i \). By Lemma 14.17 from [11], \((K_n, *)\) is a \(*\)-field.

**Theorem 4.** With the notation above, \( p_{n}(K_n, *) \leq m - 1 \) and \( ms_{n}(K_n, *) = \infty \).

**Proof.** Since \( ms_{2}(\mathbb{C}, *) = \infty \), it follows that \( ms_{2}(\mathbb{C}(z), *) = \infty \) and since \( \xi^m = 1 \), we have that \( \omega \) has order \( m \). Therefore, \( ms_{n}(K_n, *) = \infty \) by Theorem 2.

If \( a = zz^* = z^2 \cdot x^0 \) and \( b = xx^* = 1 \cdot x^2 \), then \( ab = z^2 x^2, ba = (\xi^2 z)^2 x^2 = \xi^4 z^2 x^2 \) and \( [a, b] = aba^{-1}b^{-1} = \xi^{-4} \). We assumed that \( m \neq 1, 2 \), hence \( [a, b] \neq 1 \). If \( l \) is the order of \([a, b]\), then \(-1 = [a, b] + [a, b]^2 + \ldots + [a, b]^{l-1} \) and \( l|m \). Therefore, \( p_{n}(K_n, *) \leq l - 1 \leq m - 1 \). □

**Open problem.** Fix a positive integer \( m \). Compute the minimum of \( p_{2m}(D) \) where \( D \) runs through all skew-fields such that \( ms_{2m}(D) = \infty \). Compute the minimum of \( p_{2m}(D, *) \) where \((D, \ast)\) runs through all skew-fields with involution such that \( ms_{2m}(D, \ast) = \infty \). We conjecture that both numbers are equal to 1.

**References**


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