A NOTE ON VALUES OF NONCOMMUTATIVE POLYNOMIALS

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Abstract. We find a class of algebras $A$ satisfying the following property: for every nontrivial noncommutative polynomial $f(X_1, \ldots, X_n)$, the linear span of all its values $f(a_1, \ldots, a_n)$, $a_i \in A$, equals $A$. This class includes the algebras of all bounded and all compact operators on an infinite dimensional Hilbert space.

1. Introduction

Starting with Helton's seminal paper [Hel] there has been considerable interest over the last years in values of noncommuting polynomials on matrix algebras. In one of the papers in this area the second author and Schweighofer [KS] showed that Connes’ embedding conjecture is equivalent to a certain algebraic assertion which involves the trace of polynomial values on matrices. This has motivated us [BK] to consider the linear span of values of a noncommutative polynomial $f$ on the matrix algebra $M_d(F)$; here, $F$ is a field with $\text{char}(F) = 0$. It turns out [BK Theorem 4.5] that this span can be either:

1. $\{0\}$;
2. the set of all scalar matrices;
3. the set of all trace zero matrices; or
4. the whole algebra $M_d(F)$.

From the precise statement of this theorem it also follows that if $2d > \deg f$, then (1) and (2) do not occur and (3) occurs only when $f$ is a sum of commutators.

What to expect in infinite dimensional analogues of $M_d(F)$? More specifically, let $H$ be infinite dimensional Hilbert space, and let $B(H)$ and $K(H)$ denote the algebras of all bounded and compact linear operators on $H$, respectively. What is the linear span of polynomial values in $B(H)$ and $K(H)$? A very special (but decisive, as we shall see) case of this question was settled by Halmos [Hal] and Pearcy and Topping [PT] (see also Anderson [And]) a long time ago: every operator in $B(H)$ and $K(H)$, respectively, is a sum of commutators. That is, the linear span of values of the polynomial $X_1X_2 - X_2X_1$ on $B(H)$ and $K(H)$ is the whole $B(H)$ and $K(H)$, respectively. We will prove that the same is true for every nonconstant polynomial. This result will be derived as a corollary of our main theorem which presents a class of algebras with the property that the span of values of “almost” every polynomial is equal to the whole algebra.

2. Results

By $F(X)$ we denote the free algebra over a field $F$ generated by $X = \{X_1, X_2, \ldots\}$, i.e., the algebra of all noncommutative polynomials in $X$. Let $f = f(X_1, \ldots, X_n) \in F(X)$. We say that $f$ is homogeneous in the variable $X_i$ if all monomials of $f$ have

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the same degree in $X_i$. If this degree is 1, then we say that $f$ is linear in $X_1$. If $f$ is linear in every variable $X_i$, $1 \leq i \leq n$, then we say that $f$ is multilinear.

Let $A$ be an algebra over $\mathbb{F}$. By $f(A)$ we denote the set of all values $f(a_1, \ldots, a_n)$ with $a_i \in A$, $i = 1, \ldots, n$. Recall that $f = f(X_1, \ldots, X_n) \in \mathbb{F}[X]$ is said to be an identity of $A$ if $f(A) = \{0\}$. If $f(A)$ is contained in the center of $A$, but $f$ is not an identity of $A$, then $f$ is said to be a central polynomial of $A$. By span $f(A)$ we denote the linear span of $f(A)$. We are interested in the question when does span $f(A) = A$ hold.

For the proof of our main theorem three rather elementary lemmas will be needed. The first and the simplest one is a slightly simplified version of [BK, Lemma 2.2]. Its proof is based on the standard Vandermonde argument.

**Lemma 2.1.** Let $V$ be a vector space over an infinite field $\mathbb{F}$, and let $U$ be a subspace. Suppose that $c_0, c_1, \ldots, c_n \in V$ are such that $\sum_{i=0}^n \lambda_i c_i \in U$ for all $\lambda \in \mathbb{F}$. Then each $c_i \in U$.

Recall that a vector subspace $L$ of $A$ is said to be a Lie ideal of $A$ if $[\ell, a] \in L$ for all $\ell \in L$ and $a \in A$; here, $[u, v] = uv - vu$. For a recent treatise of Lie ideals from an algebraic as well as functional analytic viewpoint we refer the reader to [BKS].

Our second lemma is a special case of [BK, Theorem 2.3].

**Lemma 2.2.** Let $A$ be an algebra over an infinite field $\mathbb{F}$, and let $f \in \mathbb{F}[X]$. Then span $f(A)$ is a Lie ideal of $A$.

Every vector subspace of the center of $A$ is obviously a Lie ideal of $A$. Lie ideals that are not contained in the center are called noncentral. The third lemma follows from an old result of Herstein [Her, Theorem 1.2].

**Lemma 2.3.** Let $S$ be a simple algebra over a field $\mathbb{F}$ with char($\mathbb{F}$) $\neq 2$. If $M$ is both a noncentral Lie ideal of $S$ and a subalgebra of $S$, then $M = S$.

We are now in a position to prove our main result.

**Theorem 2.4.** Let $S$ and $B$ be algebras over a field $\mathbb{F}$ with char($\mathbb{F}$) $= 0$, and let $A = S \otimes B$. Suppose that $S$ is simple, and suppose that $B$ satisfies

(a) every element in $B$ is a sum of commutators; and

(b) for each $n \geq 1$, every element in $B$ is a linear combination of elements $b^n$, $b \in B$.

If $f \in \mathbb{F}[X]$ is neither an identity nor a central polynomial of $S$, then

$$\text{span } f(A) = A.$$ 

**Proof.** Let $f = f(X_1, \ldots, X_n)$. Let us write $f = g_i + h_i$ where $g_i$ is a sum of all monomials of $f$ in which $X_i$ appears and $h_i$ is a sum of all monomials of $f$ in which $X_i$ does not appear. Thus, $h_i = h_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ and hence

$$h_i(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n)$$

for all $a_i \in A$. Therefore span $h_i(A) \subseteq \text{span } f(A)$, which clearly implies span $g_i(A) \subseteq$ span $f(A)$. At least one of $g_i$ and $h_i$ is neither an identity nor a central polynomial of $S$. Therefore there is no loss of generality in assuming that either $X_i$ appears in every monomial of $f$ or $f$ does not involve $X_i$ at all. Since $f$ cannot be a constant polynomial and hence it must involve some of the $X_i$’s, we may assume, again without loss of generality, that each monomial of $f$ involves all $X_i$, $i = 1, \ldots, n$.

Next we claim that there is no loss of generality in assuming that $f$ is homogeneous in $X_1$. Write $f = f_1 + \ldots + f_m$, where $f_i$ is the sum of all monomials of $f$
that have degree \( i \) in \( X_1 \). Note that
\[
f(\lambda a_1, a_2, \ldots, a_n) = \sum_{i=1}^m \lambda^i f_i(a_1, \ldots, a_n) \in \text{span } f(A)
\]
for all \( \lambda \in \mathbb{F} \) and all \( a_i \in A \), so \( f_i(a_1, \ldots, a_n) \in \text{span } f(A) \) by Lemma 2.1. Thus, \( \text{span } f_i(A) \subseteq \text{span } f(A) \). At least one \( f_i \) is neither an identity nor a central polynomial of \( S \). Therefore it suffices to prove the theorem for \( f_i \). This proves our claim.

Let us now show that there is no loss of generality in assuming that \( f \) is linear in \( X_1 \). If \( \deg_{X_1} f > 1 \), we apply the multilinearization process to \( f \), i.e., we introduce a new polynomial \( \Delta_{1,n+1} f = f'(X_1, \ldots, X_n, X_{n+1}) \):
\[
f' = f(X_1 + X_{n+1}, X_2, \ldots, X_n) - f(X_1, X_2, \ldots, X_n) - f(X_{n+1}, X_2, \ldots, X_n).
\]
This reduces the degree in \( X_1 \) by one. Clearly, \( \text{span } f'(A) \subseteq \text{span } f(A) \). Observe that \( f \) can be retrieved from \( f' \) by resubstituting \( X_{n+1} \mapsto X_1 \), more exactly
\[
(2^{\deg_{X_1}} f - 2)f = f'(X_1, \ldots, X_n, X_1).
\]
Hence \( f' \) is not an identity nor a central polynomial of \( S \). Note however that \( f' \) is not necessarily homogeneous in \( X_1 \), but for all its homogeneous components \( f'_j \) we have \( \text{span } f'_j(A) \subseteq \text{span } f'(A) \); one can check this by using Lemma 2.1 like in the previous paragraph. At least one of these components, say \( f'_j \), is not an identity nor a central polynomial of \( S \). Thus we restrict our attention to \( f'_j \). If necessary, we continue applying \( \Delta_{1,n} \) and after a finite number of steps we obtain a polynomial \( \Delta f \) linear in \( X_1 \), which is neither an identity nor a central polynomial of \( S \), and satisfies \( \text{span } \Delta f(A) \subseteq \text{span } f(A) \). Hence we may assume \( f \) is linear in \( X_1 \).

Repeating the same argument with respect to other variables we finally see that without loss of generality we may assume that \( f \) is multilinear.

Set \( L = \text{span } f(A) \) and \( M = \{ m \in S \mid m \otimes B \subseteq L \} \). By Lemma 2.2 \( L \) is a Lie ideal of \( A \). Therefore \( [m, s] \otimes b^2 = [m \otimes b, s \otimes b] \in L \) for all \( m \in M \), \( b \in B \), \( s \in S \). Using (b) it follows that \( [m, s] \in M \). Therefore \( M \) is a Lie ideal of \( S \). Pick \( s_1, \ldots, s_n \in S \) such that \( s_0 = f(s_1, \ldots, s_n) \) does not lie in the center of \( S \). For every \( b \in B \) we have
\[
s_0 \otimes b^n = f(s_1 \otimes b, s_2 \otimes b, \ldots, s_n \otimes b) \in L.
\]
In view of (b) this yields \( s_0 \in M \). Accordingly, \( M \) is a noncentral Lie ideal of \( S \). Next, given \( m \in M \) and \( b, b' \in B \), we have
\[
m^2 \otimes [b, b'] = [m \otimes b, m \otimes b'] \in L.
\]
By (a), this gives \( m^2 \in M \). From
\[
m_1 m_2 = \frac{1}{2} (m_1 m_2 + m_1 + m_2)^2 - m_1^2 - m_2^2
\]
it now follows that \( M \) is a subalgebra of \( S \). Using Lemma 2.3 we now conclude that \( M = S \), i.e., \( A = S \otimes B \subseteq L \subseteq A \).

It is easy to see that (b) is fulfilled if \( B \) has a unity. In this case the proof can be actually slightly simplified by avoiding involving powers of elements in \( B \). Further, every \( C^* \)-algebra \( B \) satisfies (b). Indeed, every element in \( B \) is a linear combination of positive elements, and for positive elements we can define rth roots.

**Corollary 2.5.** Let \( H \) be an infinite dimensional Hilbert space. Then
\[
\text{span } f(B(H)) = B(H) \quad \text{and} \quad \text{span } f(K(H)) = K(H)
\]
for every nonconstant polynomial \( f \in \mathcal{C}(\hat{X}) \).
**Proof.** It is well known that there does not exist a nonzero polynomial that was an identity of \( M_d(\mathbb{C}) \) for every \( d \geq 1 \), cf. [Row, Lemma 1.4.3]. Therefore there exists \( d \geq 1 \) such that \([f, X_{n+1}]\) is not an identity of \( M_d(\mathbb{C}) \). This means that \( f \) is neither an identity nor a central polynomial of \( M_d(\mathbb{C}) \). Since \( \mathcal{H} \) is infinite dimensional, we have \( \mathcal{B}(\mathcal{H}) \cong M_d(\mathcal{B}(\mathcal{H})) \cong M_d(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H}) \), and similarly \( \mathcal{K}(\mathcal{H}) \cong M_d(\mathbb{C}) \otimes \mathcal{K}(\mathcal{H}) \).

Now we are in a position to use Theorem 2.4. Indeed, \( M_d(\mathbb{C}) \) is a simple algebra, and the algebras \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{K}(\mathcal{H}) \) satisfy (a) by [Hal] and [PT], and they satisfy (b) by the remark preceding the statement of the corollary.

**References**


